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Likelihood ratio, optimal decision rules, and correct response probabilities in a signal detection theoretic, equal-variance Gaussian model of the observer in the 4IAX paradigm

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This article provides a synthetic account of the likelihood ratio, optimal decision rules, and correct response probabilities in a signal detection theoretic model of the observer in the *dual-pair comparison*, or *four-interval AX* (4IAX), paradigm. The model assumes a static sampling process, resulting in four, equal-variance normally distributed (i.e., Gaussian) observations on each trial. First, a likelihood ratio equation allowing for an arbitrary degree of correlation between observations is provided. Specific solutions for the cases of independent and highly correlated observations are then derived. It is shown that these solutions, and the associated decision rules, correspond to those provided independently in earlier publications. A modified 4IAX paradigm involving, as a standard, an additional stimulus (C) located medially between the A and the B stimuli is also considered. It is shown that the optimal (static, equal-variance, Gaussian) decision model for this paradigm is unaffected by correlation between observations and is equivalent to the standard 4IAX with highly correlated observations. Finally, we consider how, under the considered (static, equal-variance, Gaussian) model, the proportion of correct responses in the different versions of the 4IAX paradigm is related to d', and a solution for the case of independent observations is provided.

This article is concerned with the psychophysical paradigm known as the four-interval AX (4IAX) paradigm, the dual-pair discrimination paradigm, or the four-interval *same–different* (SD) paradigm (Creelman & Macmillan, 1979; Macmillan, Kaplan, & Creelman, 1977; Noreen, 1981; Rousseau & Ennis, 2001). In this paradigm, two pairs of stimuli are presented consecutively. One pair contains two stimuli that differ along at least one dimension—for instance, two tones of different frequency or pitch. The other pair is composed of stimuli that do not differ along that same dimension—for instance, two tones having the same frequency. The order of presentation of the two pairs is randomized, and the two possible orderings are, in general, equally likely. The subject's task is to indicate in which order the pairs were presented; typically, the instructions are to report whether the pair containing the stimuli that differed along the relevant dimension (in our example, frequency or pitch) occurred first or second. In the standard version of the 4IAX paradigm, there are two basic stimuli, A and B, and eight possible stimulus sequences: AAAB, AABA, ABAA, BAAA, BBAB, BBBA, ABBB, and BABB. This standard form of the 4IAX paradigm has been used, for instance, in speech perception studies (e.g., Beddor & Krakow, 1999; Hoffman, Daniloff, Bengoa, & Schuckers, 1985; Pisoni & Lazarus, 1974; Ramus et al., 2003). It is important to distinguish between the 4IAX paradigm, which is the object of this article, and other psychophysical paradigms, some versions of which may also involve four observation intervals. This is the case, in particular, of the four-interval oddity task (wherein three intervals contain the same stimulus, whereas the fourth interval, chosen at random, contains a different stimulus, and the observer's task is to indicate which interval contains the odd stimulus) and of the four-interval, four-alternative forced choice task. Decision models for these tasks have been discussed in detail in earlier publications (e.g., Fritjers, 1979; Versfeld, Dai, & Green, 1996).

Although decision models for the 4IAX paradigm have been considered in earlier publications (Macmillan et al., 1977; Noreen, 1981; Rousseau & Ennis, 2001), as of yet, no synthetic account has been provided. In particular, two different decision strategies have been independently proposed for this paradigm (Macmillan et al., 1977; Noreen, 1981), but the relationship between these two strategies remains unclear. The first decision strategy consists of comparing the absolute differences between the two observations in each pair and of selecting the pair for which the difference is larger (Macmillan et al., 1977); decision strategies, such as this one, that involve judgments based on differences between observations are traditionally re-

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ferred to as *differencing* strategies (Macmillan & Creelman, 1991; Noreen, 1981). An alternative was proposed by Noreen. In this alternative strategy, the observer treats each pair of observations in the 4IAX paradigm as an SD paradigm. He/she computes the likelihood ratio that the considered pair contains different stimuli and compares this likelihood ratio to that of the other pair. Finally, he/she selects the pair associated with the larger likelihood ratio. Since this strategy assumes that the different observations are statistically independent, it is traditionally referred to as the *independent observations* strategy.

A common source of correlation—and thus, of statistical dependence—between observations is the use by experimenters of across-trial stimulus roving. For instance, if in a frequency discrimination experiment in which listeners have to discriminate a tone of frequency f and a tone of frequency $f+\Delta f$ and the baseline frequency, f, is roved across trials, the observations evoked by the two tones will be correlated. The degree of correlation between observations depends on the extent of the roving; wide-roving ranges result in high correlation between observations. Dai, Versfeld, and Green (1996) have previously demonstrated that in the basic two-interval SD paradigm, the optimal likelihood ratio decision rule under a static-sampling, normally distributed, equal-variance observations signal detection theoretic model depends on the degree of correlation between the observations: Although a differencing strategy is optimal when the observations are highly correlated, when they are independent, the optimal strategy is one that takes into account the absolute position of the observations. The notion that the degree of correlation between observations can impact performance can also be understood from a pragmatic standpoint by considering that when the stimuli are roved over a wide range, real observers—whose memory capacity is limited—cannot accurately keep track of all the possible stimulus alternatives.

The present article can be thought of as an extension to the 4IAX paradigm of the type of theoretical unification achieved for the SD paradigm by Dai et al. (1996). Assuming, like many previous authors (Creelman & Macmillan, 1979; Macmillan et al., 1977; Noreen, 1981; Rousseau & Ennis, 2001), a simple static-sampling model with normally distributed equal-variance observations, we will use the notion of statistical correlation between observations to demonstrate that the two previously proposed decision strategies for the 4IAX paradigm are, in fact, particular cases of a more general likelihood ratio decision strategy, which allows for an arbitrary degree of correlation between the observations. We will demonstrate that when the correlation between observations is zero, the optimal likelihood ratio decision rule in the considered, normally distributed, equal-variance, observations model is as described by Noreen, whereas when the correlation tends toward one, the optimal decision rule under the considered model is as described by Macmillan et al.

In addition to considering the standard 4IAX paradigm, which includes two basic stimuli and eight possible stimulus sequences, we will consider a modified version of this paradigm, which involves three different stimuli (A, B, and C) but only four different stimulus sequences (CCAB, CCBA, ABCC, and BACC). The single reference stimulus, C, is intermediate between the two different stimuli (A and B) along the relevant dimension. For instance, the reference stimulus is a tone of frequency f, whereas the A and B stimuli are tones of frequencies $f+\Delta f$ and $f-\Delta f$, respectively. This modified version of the 4IAX paradigm has been used in several studies on auditory perception in recent years (e.g., Carlyon, Moore, & Micheyl, 2003; Carlyon & Shackleton, 1994; Plack & Carlyon, 1994). However, perhaps because the underlying model for this modified 4IAX paradigm had never been examined in detail until now, the results of these studies were analyzed as if they had been obtained using a 2I-2AFC paradigm. Here, we will clarify the underlying model for this modified version of the 4IAX paradigm. It will be shown that under the considered normally distributed, equal-variance observations model, the optimal decision rule for this particular version of the 4IAX paradigm is, in fact, the same as that for the standard 4IAX paradigm with highly correlated observations; a surprising outcome is that this rule does not itself depend on the degree of correlations between the observations.

Another question of potential importance for experimenters, which will also be considered in the present article, is how the optimal decision rules derived above can be used to derive predictions of the proportion of correct responses achieved by an ideal observer making optimal use of four static, normally distributed, equal-variance observations per trial in the 4IAX paradigm. In their 1977 article, Macmillan et al. provided a formula relating d' to proportion correct under the differencing strategy for the 4IAX paradigm. A different but equivalent expression involving a noncentral beta distribution was more recently derived by Rousseau and Ennis (2001). However, to our knowledge, an equation relating proportion correct to d' in the case of the 4IAX paradigm with independent (normally distributed, equal-variance) observations has not been described in earlier publications. Such an equation will be derived here.

Basic Assumptions and Model

The signal detection theory model considered here assumes that the listener makes four observations on each trial—one observation per stimulus. Each observation is modeled as a quantity, x_i (with $i = 1, 2, 3,$ or 4), proportional to the value, f_i, of a physical stimulus parameter (e.g., frequency or intensity) or some nonlinear transformation of it (e.g., level in decibels), plus a random perturbation, which reflects internal noise; formally: $x_i = \alpha f_i + n_i$. The internal noise is assumed to be normally distributed, with zero mean and a constant variance, σ^2. Each stimulus parameter value, f_i, is modeled as the sum of three components: (1) a constant, f_0, which corresponds to the reference value of the stimulus parameter (e.g., the baseline frequency in a frequency discrimination experi-

$$\lambda 4\mathrm{IAX}_{D1,D2}(Y)=\frac{\left[f_{Y|\mathrm{ABAA}}(Y\,|\mathrm{ABAA})+f_{Y|\mathrm{BAAA}}(Y\,|\mathrm{BAAA})+f_{Y|\mathrm{ABBB}}(Y\,|\mathrm{ABBB})+f_{Y|\mathrm{BABB}}(Y\,|\mathrm{BABB})\right]}{\left[f_{Y|\mathrm{AAAB}}(Y\,|\mathrm{AAAB})+f_{Y|\mathrm{AABA}}(Y\,|\mathrm{AABA})+f_{Y|\mathrm{BBAB}}(Y\,|\mathrm{BBAB})+f_{Y|\mathrm{BBBA}}(Y\,|\mathrm{BBBA})\right]} \quad (1)$$

ment); (2) a random deviation, r, which corresponds to the amount of roving applied to all four stimulus parameter values on the considered trial (on each trial, r is drawn from a Gaussian distribution with zero mean and a constant variance, σ_r^2); and (3) another deviation, Δ_i, whose value depends on the type of stimulus being presented. In the standard 4IAX paradigm, only two types of stimuli are used. They are traditionally designated by A and B. These stimuli correspond to two different stimulus parameter values, f_A and f_B, separated by a distance, Δ, and equidistant to the reference value, f_0. Thus, if $f_A > f_B$, we have $f_A = f_0 + r + \Delta/2$, and $f_B = f_0 + r - \Delta/2$. A modified version of the 4IAX paradigm, which will also be considered in this article, uses as a standard a third type of stimulus, C, the position of which along the physical continuum is located halfway between those of A and B, so that $f_C = (f_A + f_B)/2$. The full observation model is, thus, $x_i = \alpha(f_0 + r + \Delta_i) + n_i$, where Δ_i can take a value of $+\Delta/2$ (for Stimulus A) or $-\Delta/2$ (for Stimulus B) in the standard 4IAX with roving and can additionally take a value of zero (for Stimulus C) in the modified 4IAX paradigm.

To simplify the derivations, the original vector of observations, $X = [x_1, x_2, x_3, x_4]^T$ is transformed into another vector of observations, $Y = [y_1, y_2, y_3, y_4]^T$, using the following equation: $y_i = (x_i - \alpha f_0)/\sigma$. It can be shown that the variance of the transformed observations, $V[y_i]$ is $\alpha^2\sigma_r^2/\sigma^2 + 1$. For simplicity, we define $\sigma_R^2 = \alpha^2\sigma_r^2/\sigma^2$, so that, in the calculations below, $V[y_i]$ appears as $\sigma_R^2 + 1$. It can also be shown that the difference between the expected values of the transformed observations conditioned on the occurrence of Stimulus A and B, respectively—that is, $E[y_i|A] - E[y_i|B]$—equals $\alpha\Delta/\sigma$. This quantity, which we call d, is related to the index of performance from signal detection theory, d'—which is traditionally defined as the difference between the means of the internal distributions of activity evoked by the two stimuli that must be discriminated, divided by their common standard deviation (Green & Swets, 1966; Macmillan & Creelman, 1991)—by the following equation:

$$d' = \frac{d}{\sqrt{\alpha^2\sigma_r^2 / \sigma^2 + 1}}.$$

The use of the transformed variables, instead of the original ones, substantially simplifies the derivations and implies no loss of generality: The solutions that are given below in terms of the transformed variables can easily be expressed, instead, in terms of the original variables by substituting $(x_i - \alpha f_0)/\sigma$ for y_i, $\alpha^2\sigma_r^2/\sigma^2$ for σ_R^2, and

$$d'\sqrt{\alpha^2\sigma_r^2 / \sigma^2 + 1}$$

for d.

The Likelihood Ratio for the Standard 4IAX Paradigm Under the Equal-Variance Normally Distributed Static Observations Model

In the standard 4IAX paradigm, as was indicated above, there are eight possible stimulus sequences: AAAB, AABA, ABAA, BAAA, BBAB, BBBA, ABBB, and BABB. Let us define two possible events (or hypotheses), $D1$ and $D2$, which correspond to the presentation of the pair containing the different stimuli first or second, respectively. We seek to express the ratio of the likelihood, $\lambda_{D1}(Y)$, of observing the vector of observations Y given that the pair of different stimuli was presented first, to the likelihood, $\lambda_{D2}(Y)$, of observing Y given that the pair of different stimuli was presented second. Formally, this can be written as $\lambda 4\mathrm{IAX}_{D1,D2}(Y) = f_{Y|D1}(Y|D1)/f_{Y|D2}(Y|D2) \times P(D1)/P(D2)$, where $f_{Y|D1}(Y|D1)$ and $f_{Y|D2}(Y|D2)$ are the probability density functions of the observations, conditioned on events $D1$ and $D2$, and $P(D1)$ and $P(D2)$ are the a priori probabilities of events $D1$ and $D2$, respectively. Since this is almost always the case in experimental studies, we will assume in the following that the different possible stimulus sequences are all equally likely a priori and that $P(D1) = P(D2)$. With these additional assumptions, the formula for the likelihood ratio simplifies to Equation 1 at the top of this page. Each probability density function in the expression below is of the form

$$f_{Y|S}(Y\,|\,S) = \frac{1}{(2\pi)^2\left|\Sigma^2\right|}e^{-\frac{1}{2}(Y-\mu)^T\Sigma^{-2}(Y-\mu)}, \quad (2)$$

where S is the type of trial considered (S = ABAA, BAAA, AAAB, AABA, ABBB, BABB, BBAB, or BBBA). Σ^2 is the variance–covariance matrix,

$$\Sigma^2 = \begin{bmatrix} \sigma_R^2+1 & \sigma_R^2 & \sigma_R^2 & \sigma_R^2 \\ \sigma_R^2 & \sigma_R^2+1 & \sigma_R^2 & \sigma_R^2 \\ \sigma_R^2 & \sigma_R^2 & \sigma_R^2+1 & \sigma_R^2 \\ \sigma_R^2 & \sigma_R^2 & \sigma_R^2 & \sigma_R^2+1 \end{bmatrix}. \quad (3)$$

$|\Sigma^2|$ is the determinant of the variance–covariance matrix ($|\Sigma^2| = 4\sigma_R^2+1$ in this case). Σ^{-2} is the inverse of the variance–covariance matrix,

$$\Sigma^{-2} = \frac{1}{4\sigma_R^2+1}\begin{bmatrix} 3\sigma_R^2+1 & -\sigma_R^2 & -\sigma_R^2 & -\sigma_R^2 \\ -\sigma_R^2 & 3\sigma_R^2+1 & -\sigma_R^2 & -\sigma_R^2 \\ -\sigma_R^2 & -\sigma_R^2 & 3\sigma_R^2+1 & -\sigma_R^2 \\ -\sigma_R^2 & -\sigma_R^2 & -\sigma_R^2 & 3\sigma_R^2+1 \end{bmatrix}, \quad (4)$$

where (see matrix on next page)

$$\mu = \begin{cases} \left[+\frac{d}{2},+\frac{d}{2},+\frac{d}{2},-\frac{d}{2}\right]^T & \text{if } S = \text{AAAB} \\ \left[+\frac{d}{2},+\frac{d}{2},-\frac{d}{2},+\frac{d}{2}\right]^T & \text{if } S = \text{AABA} \\ \left[+\frac{d}{2},-\frac{d}{2},+\frac{d}{2},+\frac{d}{2}\right]^T & \text{if } S = \text{ABAA} \\ \left[-\frac{d}{2},+\frac{d}{2},+\frac{d}{2},+\frac{d}{2}\right]^T & \text{if } S = \text{BAAA} \\ \left[-\frac{d}{2},-\frac{d}{2},-\frac{d}{2},+\frac{d}{2}\right]^T & \text{if } S = \text{BBBA} \\ \left[-\frac{d}{2},-\frac{d}{2},+\frac{d}{2},-\frac{d}{2}\right]^T & \text{if } S = \text{BBAB} \\ \left[-\frac{d}{2},+\frac{d}{2},-\frac{d}{2},-\frac{d}{2}\right]^T & \text{if } S = \text{BABB} \\ \left[+\frac{d}{2},-\frac{d}{2},-\frac{d}{2},-\frac{d}{2}\right]^T & \text{if } S = \text{ABBB} \end{cases}.$$

After plugging Equations 2–4 into Equation 1 and simplifying, which was done with the aid of Mathematica (Wolfram Research Inc.), one finds Equation 5 below, where d is, as defined above, the unsigned difference between the means, μ_A and μ_B, of the probability density functions of the transformed observations evoked by Stimuli A and B.

Independent observations. By definition, the correlation coefficient between the observations from each trial is

$$\rho = \sqrt{\sigma_R^2 / \left(1 + \sigma_R^2\right)}.$$

When the observations are uncorrelated, $\rho = 0$ and $\sigma_R^2 = 0$. Setting σ_R^2 to zero in Equation 5, we obtain the likelihood ratio for the standard 4IAX paradigm with independent observations (4IAXio):

$$\lambda 4\text{IAXio}_{D1,D2}\left(y_1, y_2, y_3, y_4\right) = \left(\frac{e^{dy_1} + e^{dy_2}}{e^{dy_3} + e^{dy_4}}\right) \cdot \left[\frac{1 + e^{d(y_3+y_4)}}{1 + e^{d(y_1+y_2)}}\right]. \quad (6)$$

This formula is equivalent to that obtained by following Noreen's (1981, p. 275) three-step decision rule for the 4IAX paradigm with independent observations. Specifically, Noreen pointed out that the two observations from each pair in the 4IAX paradigm could be likened to the first and second (single) observations in the SD paradigm, whereas the two pairs of observations in the 4IAX paradigm could be likened to the two (single) observations in the two-interval two-alternative forced choice (2I-2AFC) paradigm. On the basis of these analogies, he proposed that the likelihood ratio for the 4IAX paradigm should correspond to the ratio of two likelihood ratios, each having the same form as the likelihood ratio for the SD paradigm, one based on the first two observations, the other on the second two observations. The optimal likelihood ratio for the independent observations strategy in the SD paradigm is, according to Noreen,[1]

$$\lambda \text{SD}_{D1,D2}\left(y_1, y_2\right) = \frac{1 + e^{d(y_1+y_2)}}{e^{dy_1} + e^{dy_2}}. \quad (7)$$

Forming the ratio of two such likelihood ratios, for two pairs of observations (y_1 and y_2, and y_3 and y_4) one obtains Equation 6. Thus, in the case of independent observations, the likelihood ratio for the 4IAX paradigm is as suggested by Noreen (1981).

Solving Equation 6 for $\lambda 4\text{IAXio}_{D1,D2}(Y) > 1$, which corresponds to responding "$D1$,"

$$\frac{1 + e^{d(y_1+y_2)}}{e^{dy_1} + e^{dy_2}} < \frac{1 + e^{d(y_3+y_4)}}{e^{dy_3} + e^{dy_4}}. \quad (8)$$

This reduces to

$$y_1 y_2 < y_3 y_4. \quad (9)$$

Conversely, for $\lambda 4\text{IAXio}_{D1,D2}(Y) < 1$, which corresponds to responding "$D2$,"

$$\frac{1 + e^{d(y_1+y_2)}}{e^{dy_1} + e^{dy_2}} > \frac{1 + e^{d(y_3+y_4)}}{e^{dy_3} + e^{dy_4}}. \quad (10)$$

This reduces to

$$y_1 y_2 > y_3 y_4. \quad (11)$$

Equations 8 and 10 reveal that the optimal decision rule in the standard 4IAX paradigm with independent observations involves a comparison between two likelihood ratios, which are derived from the first and second pairs of observations, respectively. Each of these likelihood ratios is of the same form as the likelihood ratio for the SD paradigm in the case of independent observations (Equation 7). Thus, the optimal observer in the 4IAX paradigm with independent observations is one that treats each stimulus pair as an SD trial, computes for each pair a quantity equal to (or monotonic with) the likelihood ratio given by Equation 7, then compares the two resulting values and, finally, selects the pair that is associated with the smaller value as that containing the different stimuli. From that point of view, as has been pointed out by Noreen (1981), the 4IAX paradigm can be conceived of as a 2I-2AFC paradigm with the two stimuli, A and B, replaced by *same* and *different* stimulus pairs.[2]

Highly correlated observations. If the stimuli are roved over a relatively wide range across trials, then

$$\lambda 4\text{IAX}_{D1,D2}(Y) = e^{d(y_2-y_4)} \frac{\left(1 + e^{d[y_1-y_2]}\right)\left(1 + e^{d\left[y_3+y_4+2\sigma_R^2(y_3+y_4-y_1-y_2)\right]/\left[1+4\sigma_R^2\right]}\right)}{\left(1 + e^{d[y_3-y_4]}\right)\left(1 + e^{d\left[y_1+y_2+2\sigma_R^2(y_1+y_2-y_3-y_4)\right]/\left[1+4\sigma_R^2\right]}\right)} \quad (5)$$

$\sigma_R^2 >> 1$, and $\rho \to 1$. This situation is traditionally referred to as *highly correlated observations*. The likelihood ratio for this case can be found by replacing σ_R^2 by $+\infty$ in Equation 5; it is

$$\lambda 4\text{IAXhco}_{D1,D2}\left(y_1, y_2, y_3, y_4\right) = \left(\frac{e^{dy_1} + e^{dy_2}}{e^{dy_3} + e^{dy_4}}\right)\frac{e^{\frac{d}{2}(y_3+y_4)}}{e^{\frac{d}{2}(y_1+y_2)}}. \quad (12)$$

This can be rewritten as

$$\lambda 4\text{IAXhco}_{D1,D2}\left(y_1, y_2, y_3, y_4\right) = \frac{e^{\frac{d}{2}y_1}e^{-\frac{d}{2}y_2} + e^{-\frac{d}{2}y_1}e^{\frac{d}{2}y_2}}{e^{\frac{d}{2}y_3}e^{-\frac{d}{2}y_4} + e^{-\frac{d}{2}y_3}e^{\frac{d}{2}y_4}}. \quad (13)$$

Using the fact that $\cosh(z) = (e^z + e^{-z})/2$, Equation 13 can be reexpressed as

$$\lambda 4\text{IAXhco}_{D1,D2}\left(y_1, y_2, y_3, y_4\right) = \frac{\cosh\left[\frac{d}{2}\left(y_1 - y_2\right)\right]}{\cosh\left[\frac{d}{2}\left(y_3 - y_4\right)\right]}. \quad (14)$$

Solving Equation 14 for $\lambda 4\text{IAXhco}_{D1,D2}(Y) > 1$, we find[3]

$$|y_1 - y_2| > |y_3 - y_4|. \quad (15)$$

Solving Equation 14 for $\lambda 4\text{IAXhco}_{D1,D2}(Y) < 1$, we find

$$|y_1 - y_2| < |y_3 - y_4|. \quad (16)$$

Thus, when the observations are correlated, the optimal (likelihood ratio) decision rule in the 4IAX paradigm with a single, median standard is to compare the absolute differences between the two observations from each pair. This is the decision rule suggested by Macmillan et al. (1977).

The Likelihood Ratio for the Modified 4IAX Paradigm Under the Equal-Variance Normally Distributed Static Observations Model

In the past decade, several studies of auditory perception (e.g., Carlyon & Shackleton, 1994; Plack & Carlyon, 1994) have involved experiments in which listeners were asked to indicate which of two consecutive pairs of simultaneous stimuli occupying different frequency regions contained stimuli that differed along one parameter, such as level (Dai & Green, 1992), fundamental frequency (Carlyon & Shackleton, 1994), frequency modulation depth (Plack & Carlyon, 1994), frequency modulation phase (Carlyon et al., 2003), or amplitude modulation phase (Strickland, Viemeister, Fantini, & Garrison, 1989). If it is assumed that listeners made at least one observation for each stimulus, those experiments can be thought of as involving a modified 4IAX paradigm, which uses a single reference stimulus, C, whose position on the relevant physical or sensory continuum is halfway between the A and the B stimuli. In most, albeit not all, of the studies cited above, the stimuli were roved widely along the relevant continuum across trials.

Because of the use of a single standard, the likelihood ratio for this modified 4IAX paradigm contains only two terms in its numerator and two in its denominator:

$$\lambda 4\text{IAXm}_{D1,D2}\left(Y\right) = \frac{f_Y\left(Y\,|\text{ABCC}\right) + f_Y\left(Y\,|\text{BACC}\right)}{f_Y\left(Y\,|\text{CCAB}\right) + f_Y\left(Y\,|\text{CCBA}\right)}, \quad (17)$$

and the mean vector, μ, can take one of only four possible sets of values—namely, $\mu = [0, 0, +d/2, -d/2]^T$ if the presented stimulus, S, equals CCAB, $\mu = [0, 0, -d/2, +d/2]^T$ if S = CCBA, $\mu = [+d/2, -d/2, 0, 0]^T$ if S = ABCC, and $\mu = [-d/2, +d/2, 0, 0]^T$ if S = BACC. Apart from this, the same derivations as those above (Equations 2–4) can be used in order to find the fully developed likelihood ratio for this case:

$$\lambda 4\text{IAXm}_{D1,D2}\left(Y\right) = \left(\frac{e^{dy_1} + e^{dy_2}}{e^{dy_3} + e^{dy_4}}\right)\left[\frac{e^{\frac{d}{2}(y_3+y_4)}}{e^{\frac{d}{2}(y_1+y_2)}}\right]. \quad (18)$$

The right-hand side of this equation is the same as that of Equation 12. Thus, the likelihood ratio and the optimal (likelihood ratio) decision rule in this modified 4IAX paradigm are the same as those in the standard 4IAX paradigm with highly correlated observations. It is worth noting that σ_R^2 does not appear in this equation. This indicates that in this modified 4IAX paradigm, the optimal (likelihood ratio) decision rule is unaffected by the degree of correlation between the observations. This outcome may seem counterintuitive. Indeed, given the seemingly slight difference between the standard and the modified 4IAX paradigms, it is surprising that performance depends so much on roving in the former, but not in the latter. Some intuition as to why this is the case can be gained by noting that in the standard 4IAX paradigm with independent observations, the likelihood ratio equation (Equation 6) involves sums between the observations in each pair ($y_1 + y_2$ and $y_3 + y_4$). The readers can easily convince themselves that the expected value of these sums will differ depending on whether the stimuli that gave rise to the corresponding observations were identical or different. Thus, the observer can benefit from comparing the value of the two considered sums to some fixed internal reference—corresponding, for instance, to the expected value of the four observations. Obviously, if the stimuli are widely roved, the values taken by the considered sums will vary tremendously around the expected value. The reason the sums are not useful in the modified 4IAX paradigm even when the observations are independent is simply that since the A and B stimuli are now equidistant to the reference C stimulus, the expected value of the sum of the observations from each trial is the same regardless of whether the two presented stimuli are identical or different.

One implication of this result, according to which, in the equal variance Gaussian observations model considered here, the optimal decision rule for the modified 4IAX paradigm does not depend on the degree of correlation between observations, is that if this model is correct, performance in the modified 4IAX should be unaffected by roving.[4] It will be interesting to see whether this prediction is met by experimental results. If the experimental results occasionally fail to conform to this prediction, it may provide insight into when and why observers do not behave like the optimal likelihood ratio observers considered here. If the discrepancy is more systematic, it would call into question the validity of the simple signal detection theoretic model for the 4IAX paradigm considered in this article and lead to the formulation of more elaborate models in which performance can be affected by roving. One way of achieving this would be, as Durlach and Braida (1969) did, to add into the model an additional source of internal noise, the magnitude of which depends on the number of possible stimuli (or the range of stimulus variation) used in the experiment. The outcome that roving is not required in principle is also interesting from a practical standpoint. By contributing to raise the observer's uncertainty, the use of roving may, in some cases, have undesirable consequences on the measurement of performance. The present results suggest that in such cases, the modified 4IAX paradigm with a single, intermediate standard may be more suited than the standard 4IAX paradigm with two possible standards. This will have to be checked empirically.

The Relationship Between *d′* and Proportion Correct in the 4IAX Paradigm Under the Equal-Variance Normally Distributed Static Observations Model

An important question for the experimenter is the following: What proportion of correct responses will be achieved by an observer following the above-described optimal decision rule in the different versions of the 4IAX paradigm considered above? Clearly, a correct response will occur each time that the pair indicated by the observer happens to contain the two different stimuli. In other words, a correct response will occur each time the observer responds "first pair" and the different stimuli were presented in the first pair or the observer responds "second pair" and the different stimuli were presented in the second pair. The response, "first pair" or "second pair," is determined entirely by the observations and the decision rule. Therefore, knowing how the observations are distributed and knowing the decision rule, we can compute the probability that a given stimulus will be correctly responded to. This probability is the integral, over the region that corresponds to the correct decision, of the internal probability density function evoked by the considered stimulus.[5] Formally,

$$Pc = \iiiint_R f_{Y|\mathrm{ABAA}}(y_1, y_2, y_3, y_4 \mid \mathrm{ABAA})\, dy_1\, dy_2\, dy_3\, dy_4, \quad (19)$$

where R is the region of the decision space over which, in this case, the likelihood ratio exceeds one. The main challenge in carrying out the integration properly is in defining the integration boundaries.

In the case of highly correlated observations, the solution is relatively straightforward. It was published by Macmillan et al. (1977) and has since then appeared in other articles (e.g., Rousseau & Ennis, 2001). It is again included and briefly discussed below, for completeness. For the case of independent observations in the standard 4IAX design, the solution is less straightforward and, to our knowledge, has not been provided in earlier publications. It will be given below.

Standard 4IAX paradigm with highly correlated observations. A formula giving the proportion of correct responses as a function of d' for an optimal observer in the standard 4IAX paradigm with highly correlated observations was provided by Macmillan et al. (1977). It is

$$Pc = \left[\Phi\left(\frac{d'}{2}\right)\right]^2 + \left[1 - \Phi\left(\frac{d'}{2}\right)\right]^2, \quad (20)$$

where Φ denotes the normal integral. An alternative formulation, involving a singly noncentral beta distribution, was more recently described by Rousseau and Ennis (2001). The alternative formulation has the computational advantage of involving a single integral and constant limits of integration but is otherwise equivalent to the more traditional—and perhaps, more intuitively appealing—formulation initially provided by Macmillan et al. It is worth pointing out that Equation 20 is equivalent to the equation provided by Noreen (1981) for an observer using the independent observations strategy in the SD paradigm. Thus, the relation between proportion correct and d' for an observer using the differencing strategy in the 4IAX paradigm is the same as that for an observer using the optimal independent observations strategy in the SD paradigm.

The inverse relationship, giving d' as a function of Pc for the 4IAX paradigm with highly correlated observations, can be found by completing the square in Equation 20, yielding[6]

$$d' = 2\Phi^{-1}\left[\sqrt{\frac{Pc}{2} - \frac{1}{4}} + \frac{1}{2}\right], \quad (21)$$

where Φ^{-1} denotes the inverse of the normal integral.

Some insight into how Equation 20 relates to the optimal decision rule for the standard 4IAX with highly correlated observations (Equations 15–16) can be obtained by following Macmillan et al.'s (1977) argument. This argument is cast in geometrical terms. It is based on the notion that when the observations are represented in a two-dimensional space whose axes correspond to the differences between the observations from each pair ($x_1 - x_2$ and $x_3 - x_4$), after a 45° rotation, the two decision regions defined by Equations 15 and 16 simply correspond to the upper-right and lower-left quadrants if the likelihood ratio of Equation 14 is smaller than one (and Equation 16 is true) or to the lower-right and upper-left quadrants if the likelihood ratio of Equation 14 is larger than one (and Equation 15 is true). The probability of a correct response equals the integral, over the region defined by Equation 15, of the probability density function of the differences between observations, conditioned on $D1$—the event that the different stimuli were in the first pair.

Modified 4IAX paradigm. The same geometrical argument can be used to show that Equations 20 and 21 also apply in the case of the modified 4IAX paradigm with a single, intermediate standard. Indeed, although there are only four possible stimulus alternatives in this case, the probability density functions in the $(x_1 - x_2) \times (x_3 - x_4)$ decision space have the same shape and relative positions as those pictured by Macmillan et al. (1977). The readers can easily convince themselves that this is the case by realizing that although the set of possible stimulus sequences in the modified 4IAX paradigm is different from that in the standard 4IAX paradigm, the sets of possible *differences* between the observations from each pair are identical. Furthermore, we have proven above that the optimal decision rule for the modified 4IAX paradigm is the same as that for standard 4IAX paradigm with highly correlated observations. Therefore, the decision regions are also the same.

Standard 4IAX paradigm with independent observations. One case not covered in the previous two paragraphs is that of the standard 4IAX paradigm with independent observations. In this case, the relationship between proportion correct and d' is not as straightforward, because the original four-dimensional decision space cannot, without some loss of information, be reduced to a two-dimensional space having as axes the differences between the two observations from each pair. One approach that can be used to reduce the complexity of the problem[7] is to first fix the product of the observations from one pair—say, $y_1 y_2$—at a certain value, c, and then to find the boundaries of the region(s) of the $y_3 \times y_4$ space over which $y_3 y_4 > c$. The region(s) in question correspond(s) to the case where the response is "first pair"; thus, all combinations of observations falling into the considered region(s) will yield a correct response from the optimal observer if the first pair contains the different stimuli. Once y_1 and y_2 are fixed, y_3 can still take any value in the range from $-\infty$ to $+\infty$, but the value of y_4 is limited by the constraint that $y_3 y_4 > c$, or equivalently, $y_4 > c/y_3$. There are then two cases, depending on the sign of c, as illustrated in Figure 1. For $c > 0$, we have $(y_1 y_2)/y_3 < y_4 < \infty$ when $y_3 > 0$, and $-\infty < y_4 < (y_1 y_2)/y_3$ when $y_3 < 0$. The corresponding correct decision region is shown in the upper panel of Figure 1. For $c < 0$, we have $(y_1 y_2)/y_3 < y_4 < \infty$ when $y_3 > 0$, and $-\infty < y_4 < (y_1 y_2)/y_3$ when $y_3 < 0$. The corresponding correct decision region is shown in the lower panel of Figure 1.

Since the limits of integration for the two considered cases ($c > 0$ and $c < 0$) are identical, the equation for the probability of a correct response simplifies to

$$Pc = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[\int_{0}^{+\infty}\int_{\frac{y_1 y_2}{y_3}}^{+\infty} f_{Y|\mathrm{ABAA}}(Y\,|\,\mathrm{ABAA})\,dy_4 dy_3 + \int_{-\infty}^{0}\int_{-\infty}^{\frac{y_1 y_2}{y_3}} f_{Y|\mathrm{ABAA}}(Y\,|\,\mathrm{ABAA})\,dy_4 dy_3\right] dy_2 dy_1, \quad (22)$$

where $f_{Y|\mathrm{ABAA}}(Y|\mathrm{ABAA})$ is as defined above (in relation to Equation 1).[8] Since, in the case considered here, the observations are independent, Equation 22 expands to Equation 23 (see on following page).

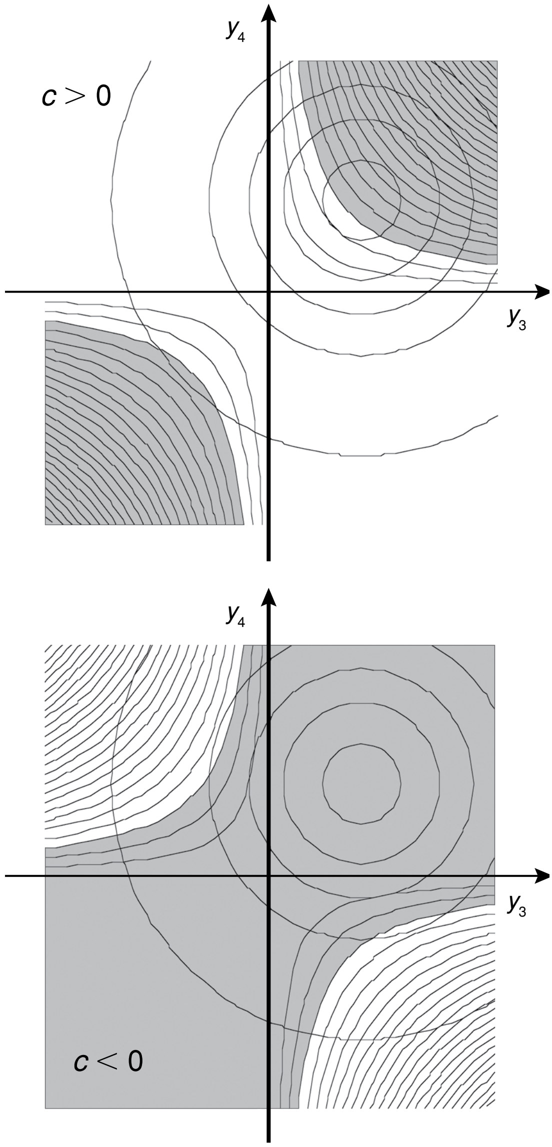

Figure 1. A subset of the decision space for the four-interval AX (4IAX) paradigm with independent observations. Because there are four independent observations, the complete decision space for this paradigm is four-dimensional. This figure shows only a two-dimensional slice through this four-dimensional space. The coordinates of points falling in this plane are the observations y_3 and y_4, corresponding to the second stimulus pair. The other two observations, y_1 and y_2, are assumed to be fixed. The circular contours correspond to equidistant equal-probability contours on the bivariate Gaussian probability density function of the y_3 and y_4 observations, assuming that the ABAA stimulus is presented. The other curves show contours corresponding to constant values of the $y_1 y_2$ product, c. Two cases are considered: The case in which $c > 0$ is shown in the upper panel; the case in which $c < 0$ is shown in the lower panel. The shaded areas show regions over which $y_3 y_4 > y_1 y_2$ in these two cases.

$$Pc = \int_{-\infty}^{+\infty} f_{y_1|\mathrm{ABAA}}\left(y_1 \mid \mathrm{ABAA}\right) \int_{-\infty}^{+\infty} f_{y_2|\mathrm{ABAA}}\left(y_2 \mid \mathrm{ABAA}\right) \left[\int_{0}^{+\infty} f_{y_3|\mathrm{ABAA}}\left(y_3 \mid \mathrm{ABAA}\right) \int_{\frac{y_1 y_2}{y_3}}^{+\infty} f_{y_4|\mathrm{ABAA}}\left(y_4 \mid \mathrm{ABAA}\right) dy_4 dy_3 + \int_{-\infty}^{0} f_{y_3|\mathrm{ABAA}}\left(y_3 \mid \mathrm{ABAA}\right) \int_{-\infty}^{\frac{y_1 y_2}{y_3}} f_{y_4|\mathrm{ABAA}}\left(y_4 \mid \mathrm{ABAA}\right) dy_4 dy_3 \right] dy_2 dy_1 \quad (23)$$

Finally, assuming Gaussian probability density functions for the individual observations,

$$Pc = \int_{-\infty}^{+\infty} \phi(t) \int_{-\infty}^{+\infty} \phi(u) \left\{ \int_{0}^{+\infty} \phi(v)\left[1-\Phi(w)\right] dy_3 + \int_{-\infty}^{0} \phi(v)\,\Phi(w) dy_3 \right\} dy_2 dy_1, \quad (24)$$

where

$$t = y_1 - \frac{d}{2},$$

$$u = y_2 + \frac{d}{2},$$

$$v = y_3 - \frac{d}{2},$$

and

$$w = \frac{y_1 y_2}{y_3} - \frac{d}{2}.$$

Although Equation 24 is expressed here in terms of d, rather than d', it turns out that since—as was indicated in the first section of the article—

$$d' = d \Big/ \sqrt{\alpha^2 \sigma_r^2 / \sigma^2 + 1},$$

when the observations are independent and $\sigma_r^2 = 0$, the term under the square-root sign is just one.

Figure 2 illustrates how the proportion of correct responses defined by Equation 24 increases with d'. For comparison, we have also plotted in this figure the relationship between proportion correct and d' for the 4IAX paradigm with highly correlated observations, along with the curves corresponding to other traditional paradigms, such as yes–no, 2I-2AFC, and SD. It can be seen that the proportion of correct responses in the 4IAX paradigm always falls between that for the 2I-2AFC paradigm and that for the SD paradigm. Furthermore, the proportion of correct responses for the standard 4IAX paradigm with independent observations is larger than that for the standard 4IAX paradigm with highly correlated observations. This confirms Macmillan et al.'s (1977) intuition that unless the observations are highly correlated, the differencing strategy, which consists of comparing the differences between the two observations from each pair, is suboptimal. Another point worth noting regarding Figure 1 is that the function relating proportion correct to d' for the standard 4IAX paradigm with highly correlated observations, which is shown in this figure, is identical to the function relating proportion correct to d' in the modified 4IAX paradigm (the reason for this has been explained above; see the section entitled "The Likelihood Ratio for the Modified 4IAX Paradigm"). It is also identical to the function relating proportion correct to d' for the SD paradigm with independent observations, a formula for which has been provided in earlier publications (Dai et al., 1996; Noreen, 1981).

For readers interested in numerical values, the proportions of correct responses that correspond, in the standard 4IAX paradigm with independent observations, to d' values between 0 and 6 in steps of 0.01 are listed in Table 1. The proportions of correct responses for the standard 4IAX paradigm with highly correlated observations, which also correspond to the modified 4IAX paradigm with a median standard, are listed in Rousseau and Ennis (2001).

Summary and Limitations of the Present Work

In this article, we have derived a general equation for the likelihood ratio in a static, equal-variance, normally distributed observations model of the 4IAX paradigm, allowing for an arbitrary degree of correlation between the observations. We have shown that for the standard version of this paradigm, which involves two stimuli (A and B) and eight possible stimulus sequences, if the observations are independent, this equation simplifies into a ratio of two likelihood ratios based on the first and second observation pairs. This is consistent with the strategy described by Noreen (1981). In the case of the standard 4IAX paradigm with highly correlated observations, we showed that the likelihood ratio equation simplifies into a ratio of two even functions (cosh) having for argument a quantity proportional to the difference between observations. Solving this equation for a likelihood ratio larger than one, we proved that an optimal decision rule in this case is to compare the unsigned differences between observations from the first and the second pairs and to choose the pair

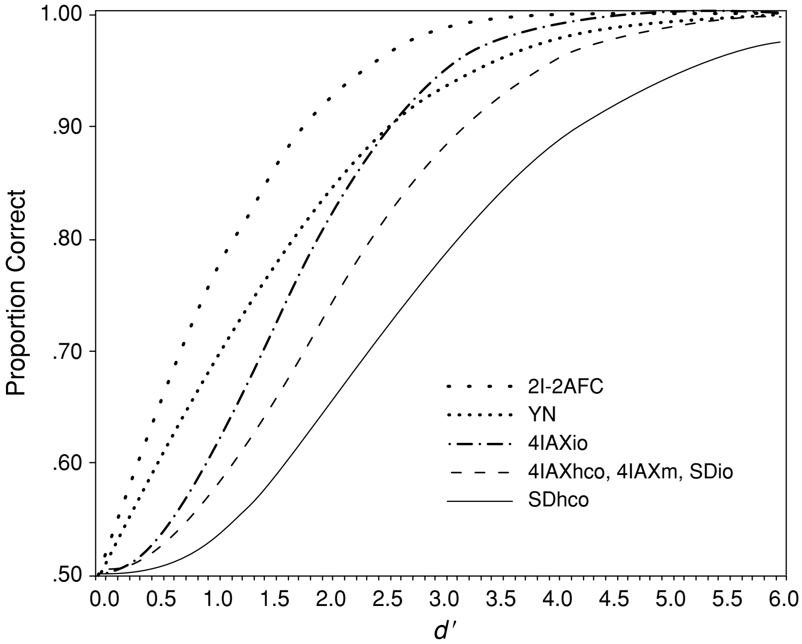

Figure 2. Proportion correct as a function of *d′* in the four-interval AX (4IAX) and other psychophysical paradigms for an unbiased observer. Far-dotted line, two-interval two-alternative forced choice (2I-2AFC) paradigm; close-dotted line, yes/no (YN) paradigm; dashed-and-dotted line, standard 4IAX with independent observations (4IAXio) paradigm; dashed line, standard 4IAX with highly correlated observations (4IAXhco) paradigm, modified 4IAX paradigm with an intermediate standard (4IAXm), or same–different with independent observations (SDio) paradigm (for these three paradigms, the relationship between proportion correct and *d′* is the same); solid line, same–different with highly correlated observations (SDhco) paradigm.

associated with the larger difference. This is the differencing strategy for the 4IAX paradigm, originally suggested by Macmillan et al. (1977). We also considered a modified version of the 4IAX paradigm, which uses as standard an additional stimulus, C, located medially between Stimuli A and B along the sensory continuum. We found that for this modified paradigm, irrespective of the degree of correlation between observations, the optimal (likelihood ratio) decision rule is the same as that for the standard 4IAX paradigm with highly correlated observations. Finally, we showed how predicted proportions of correct responses for an ideal observer in the different versions of the 4IAX paradigm can be derived by integrating the conditional (Gaussian) probability density functions of the observations over regions of the observation space defined by the optimal decision rule. This led to the conclusion that the equation relating proportion correct to d' in the modified 4IAX paradigm is the same as that for the standard 4IAX paradigm with highly correlated observations. It also led to the derivation of a new solution for the case of independent observations, which provides experimenters with a way of relating proportion correct to d' in 4IAX experiments in which the stimuli are not roved.

The limitations of the present theoretical contribution include the fact that the assumptions that were made, and the simple models that were considered, ignore the possibility of context effects or other temporal-spatial interactions between stimuli within or across trials; future developments of this work may be able to take such phenomena into account. The present work also ignores some recent developments of signal detection theory, which take into account the dynamic nature of the sensory information sampling process (e.g., Balakrishnan & MacDonald, 2003). Finally, it is important to acknowledge that in recent years, the basic assumptions of signal detection theory have been questioned (e.g., Balakrishnan, 1998, 1999; Dzhafarov, 2003a, 2003b). The present theoretical contribution obviously says nothing of the validity of the theory. Studies involving detailed comparisons between predictions from the models described here and actual psychophysical data collected in experiments in which the 4IAX paradigm has been used are needed to determine whether the assumptions underlying the considered models are valid or not. Such comparisons between model predictions and data have recently started to appear in the literature (Micheyl & Oxenham, 2005).

Table 1
Proportion of Correct Responses, *Pc*, as a Function of *d′* for the Optimal Observer in the 4IAX Paradigm With Independent Observations

d′	*Pc*	*d′*	*Pc*
0.0	.50	2.6	.91
0.1	.50	2.7	.92
0.2	.50	2.8	.93
0.3	.51	2.9	.94
0.4	.52	3.0	.95
0.5	.53	3.1	.95
0.6	.54	3.2	.96
0.7	.56	3.3	.97
0.8	.57	3.4	.97
0.9	.59	3.5	.98
1.0	.61	3.6	.98
1.1	.63	3.7	.98
1.2	.65	3.8	.99
1.3	.67	3.9	.99
1.4	.69	4.0	.99
1.5	.71	4.1	.99
1.6	.73	4.2	.99
1.7	.76	4.3	.99
1.8	.78	4.4	1.00
1.9	.80	4.5	1.00
2.0	.81	4.6	1.00
2.1	.83	4.7	1.00
2.2	.85	4.8	1.00
2.3	.87	4.9	1.00
2.4	.88	5.0	1.00
2.5	.89	>5.0	1.00

Note—The *Pc* values were obtained by numerical evaluation of Equation 24 under MATLAB (The MathWorks Inc.) using Newton's method with an integration step size of 0.01. The infinite integration boundaries in Equation 24 were replaced by $d'-6$ and $d'+6$, respectively. The MATLAB code used to carry out these computations is available, upon request, from the first author (cmicheyl@mit.edu).

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NOTES

1. In Noreen's (1981) article, the likelihood ratio for the SD paradigm with independent observations is the inverse of that indicated in our Equation 7. This should be kept in mind when comparing Noreen's description of the optimal decision rule in the 4IAX paradigm with that presented here. If the nominator and the denominator in the likelihood ratio given by Equation 7 are swapped, the decision rule should be inverted. We chose Equation 7 here to be consistent with the article by Dai, Versfeld, and Green (1996). The present Equation 7 is equivalent to Equation 10 in that article.

2. This superficial analogy between the 4IAX and the 2I-2AFC paradigms should not be interpreted to mean that the experimental results obtained with the former can be treated simply as if they had been obtained with the latter. In particular, as will be illustrated below, the relationship

between proportion correct and d' is quite different in the two paradigms. This is worth stressing because, in earlier studies, experimental data collected with a 4IAX paradigm have been erroneously interpreted as if they had been obtained with a 2I-2AFC paradigm, with important theoretical implications (Micheyl & Oxenham, 2005).

3. The presence of the absolute value operator in this and the next solution comes from the fact that the hyperbolic cosine function is even.

4. Another situation in which roving should have no influence on performance in the 4IAX paradigm is one in which listeners base their decisions on just one stimulus pair, instead of using the observations from both pairs. However, in this case, they should perform worse than predicted, and their performance in the standard 4IAX paradigm should also be independent of whether the stimuli are roved or not.

5. If the observer is unbiased (i.e., equally prone a priori to choose the first pair or the second pair) and the different stimuli are equally likely to occur on the first pair or the second pair, the probability of responding "first pair" while the different stimuli are presented in the first pair should equal the probability of responding "second pair" while the different stimuli are presented in the second pair. Consequently, we only have to compute one of these two probabilities in order to obtain the probability of a correct response. Here, we choose to compute the probability of correct responses for the situation in which the different stimuli are in the first pair. Given this situation, there are four possible stimuli (ABAA, BAAA, ABBB, and BABB). However, because the different stimuli are equally likely and the probabilities correct conditioned on any given stimulus are equal (by symmetry of the joint probability function), we may solve for only one of the four stimulus cases. This is done for ABAA, as an example. The equation for the proportion correct in the case in which the different stimuli occur in the second pair is similar to Equation 24, except that the integration boundaries of y_3 are swapped between the first and the second terms in brackets. This can be understood by considering that the decision space for that case is the same as that for the case treated here, after 90º rotation.

6. Although Equation 18 can be derived relatively straightforwardly by completing the square in Equation 17, we provide the solution here because the solution indicated in an earlier publication (Rousseau & Ennis, 2001, Equation 6) was inaccurate.

7. The solution might be arrived at by using a decision space that has for coordinates the products of the observations from each pair—that is, the $y_1y_2 \times y_3y_4$ space. This is not the approach that we have used here.

8. Note that the first integral within the brackets is for $y_3 < 0$, whereas the second integral is for $y_3 > 0$; the inner integral is undefined when $y_3 = 0$.

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